Mon. Not. R. Astron. Soc. **000**, 1–13 () Printed 21 September 2009 (MN LaTeX style file v2.2)

Nuclear physics of reverse electron flow at pulsar polar caps

P. B. Jones*
Department of Physics, University of Oxford, Denys Wilkinson Building, Keble Road, Oxford, OX1 3RH, England

ABSTRACT
Protons produced in electromagnetic showers formed by the reverse-electron flux are usually the largest component of the time-averaged polar-cap open magnetic-flux line current in neutron stars with positive corotational charge density. Although the electric-field boundary conditions in the corotating frame are time-independent, instabilities on both medium and short time-scales cause the current to alternate between states in which either protons or positrons and ions form the major component. These properties are briefly discussed in relation to nulling and microstructure in radio pulsars, pair production in an outer gap, and neutron stars with high surface temperatures.

Key words: pulsars: general - stars: neutron - plasmas - instabilities

1 INTRODUCTION

Polar-cap models of pulsar radio and X-ray emission all assume electron-positron pair creation by particles accelerated along the narrow bundle of open magnetic flux lines that intersect the light cylinder. Understanding of the radio emission process remains limited. In a review of the subject, Melrose (1995) made the interesting observation that its characteristics are broadly unchanged over more than five orders of magnitude variation in magnetic dipole moment. He also noted that whilst instabilities in the electron-positron plasma received much attention, there had been relatively less interest in the origin of the plasma. But considerable progress with the latter problem has been made by Muslimov & Tsygan (1992) who recognized that the Lense-Thirring effect, the general-relativistic dragging of inertial frames, has a profound influence on the acceleration electric field, which is the component $\mathbf{E}_{\parallel}$ locally parallel with $\mathbf{B}$, present in the frame of reference rotating with the neutron star. The electric field is conservative in the rotating frame and is given in terms of the charge density ρ by,

$$\nabla \cdot \mathbf{E} = 4\pi\left(\rho - \rho_0\right), \tag{1}$$

where, in Euclidean space,

$$\rho_0 = -\frac{1}{4\pi c}\nabla \cdot \left((\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}\right), \tag{2}$$

is the density at which force-free corotation of the magnetosphere exists (Goldreich & Julian 1969). The most significant way in which a general-relativistic treatment (see Muslimov & Harding 1997; equations 37 and 38) modifies equation (2) is that the rotation angular velocity $\mathbf{\Omega}$ seen by a distant observer is replaced by $\mathbf{\Omega} - \mathbf{\Omega}_{\rm LT}$, where $\mathbf{\Omega}_{\rm LT}$ is the Lense-Thirring angular velocity. In the non-relativistic equation (1), both ρ and ρ_0 have the same radial dependence because the angle between $\mathbf{\Omega}$ and $\mathbf{B}$ does not change much over distances no more than an order of magnitude greater than the polar cap radius. But ρ and ρ_0 have different physical origins, ρ being determined principally by continuity, so that the presence of the Lense-Thirring angular velocity

$$\mathbf{\Omega}_{\rm LT} = \frac{2G}{c^2 r^3}\mathbf{L}, \tag{3}$$

where $\mathbf{L}$ is the angular momentum of the star, in the general-relativistic expression for ρ_0 produces a non-trivial radial dependence of the quantity $\rho - \rho_0$ which is the source of the acceleration field.

In the corotating frame, the potential boundary condition for equation (1) is $\Phi = 0$ on the neutron star surface and on the surface separating open from closed magnetic flux lines, with $\mathbf{E} = -\nabla\Phi$. This is supplemented by the condition $\mathbf{E}_{\parallel} = 0$ on the neutron star surface if the work function there is not large enough to affect the free flow of charge. Thus there are three possible cases: (i) $\mathbf{\Omega} \cdot \mathbf{B} > 0$, $\rho_0 < 0$, $\mathbf{E}_{\parallel} = 0$ with electron acceleration; (ii) $\mathbf{\Omega} \cdot \mathbf{B} < 0$, $\rho_0 > 0$, $\mathbf{E}_{\parallel} = 0$, giving ion and positron acceleration or (iii), $\mathbf{\Omega} \cdot \mathbf{B} < 0$ with $\mathbf{E}_{\parallel} \neq 0$. The latter case is that considered originally by Ruderman & Sutherland (1975) in which there is now renewed interest (see Gil et al 2008, and work cited therein) owing to the possibility that some neutron-star polar-cap magnetic fields may exceed the critical field $B_c = m^2c^3/e\hbar = 4.41 \times 10^{13}$ G and may even be one or more orders of magnitude larger than the dipole moment inferred from spin-down indicates. Recent calculations of ionic

* E-mail: p.jones1@physics.ox.ac.uk

work functions (Medin & Lai 2006), including magnetic flux densities $B > 10^{14}$ G, do not unequivocally exclude the neutron-star surface boundary condition $\mathbf{E}_{\parallel} \neq 0$.

Following the work of Muslimov & Tsygan, case (i) appears consistent with electron acceleration and pair production occurring over a very wide range of surface magnetic flux densities. Ion acceleration occurs in case (ii) with the possibility of positron production by secondary processes. However, the time-independent boundary conditions $\Phi = 0$ and $\mathbf{E}_{\parallel} = 0$ applying in cases (i) and (ii) are very restrictive and in terms of electromagnetic considerations alone there is no obvious reason why either the current density or the nature of the electron-positron plasma within the open magnetic flux-line region should change so as to produce phenomena such as pulse nulling over times of the order of $10^1 - 10^5$s. (For example, failure of the condition $\Phi = 0$ on the surface separating open from closed magnetic flux lines over such long time intervals appears unlikely.) In this connection, it is worth noting that in PSR B1931+24, nulls last for intervals so long that it has been possible to measure the spin-down rate separately during the on and off-states of emission (Kramer et al 2006). Spin-down in the on-state is approximately twice as fast as in the off-states. In each of cases (i) and (ii), the self-consistent current density in the open magnetic flux-line region (see Muslimov & Harding 1997) does not deviate by so large a factor from the Goldreich-Julian value of $\rho_0 c$. Thus some change in the nature of the plasma is indicated. Also there is the fact that the radio emission from many pulsars exhibits short-term variability in the form of both microstructure and subpulses. The microstructure may be caused by temporal variation of the accelerated plasma or may be a consequence of angular beaming, but there appears to be as yet no conscensus (see, for example, Smirnova et al 1994 and Lange et al 1998). It is not obvious how any of these phenomena arise from electron or positron acceleration subject to time-independent boundary conditions. For these reasons, the present paper addresses the question of whether or not it is plausible that case (iii) actually exists in certain radio pulsars and, specifically, to an investigation of nuclear processes in the electromagnetic showers produced by the reverse flow of electrons toward the polar cap. In the course of this, we also find that case (ii) is actually less simple than might be supposed and that, even in the presence of time-independent boundary conditions, natural instabilities exist that cause medium and short time-scale variability in the nature of the accelerated plasma.

Nuclear processes were considered previously at magnetic flux densities $B \ll B_c$ in a less than satisfactory attempt to estimate backward photon production by the electromagnetic showers that are formed at the neutron-star surface (Jones 1979). But the present work is concerned with $B \sim B_c$ because only in this region is there a possibility that case (iii) could exist. Also the problem here is much more simple: that of finding the rate of photo-production of protons by decay of the nuclear giant-dipole resonance (GDR), for which experimental data has greatly improved. The mean electron-shower energy required to create a proton is a function of B and of the nuclear charge Z, but at $B > B_c$ is found to be no more than of the order of $2 - 5$ GeV. Diffusion to the neutron-star surface occurs with a short, but important, time delay and the process leads directly to an upper limit for a steady-state surface blackbody temperature in cases (ii) and (iii). But further analysis of the short-time instability in case (iii) shows that T_{res}, the polar-cap temperature in the absence of a reverse-electron flux, is actually the temperature which should be compared with the critical temperature T_c inferred from calculated ionic work functions. The details of shower development and proton formation and diffusion leading to these conclusions are given in Sections 2 and 3. The application to pulsar polar caps is described in Section 4.

Proton production by the reverse flux of electrons is prolific even at $B \ll B_c$ (see Section 2.5) and in Section 4.3 it is shown that it generally leads to medium time-scale instability in cases (ii) and (iii) producing similar variabilty in electron-positron pair formation. Instability with a short time-scale is also likely to be a property of cases (ii) and (iii) but unfortunately, is not possible to predict with complete confidence because it depends on details of the proton diffusion to the polar-cap surface that are not known with certainty. However, it is of some interest in view of the observed characteristics of individual radio pulses in some pulsars.

The principal results of this paper are the proton formation rates at $B \sim B_c$ and the instabilities found in the fluxes of accelerated nuclei and protons and in pair formation. Section 5 considers briefly the possibility that these may be connected with pulse nulling and microstructure, pair formation in any outer-magnetosphic gap, and the radio-observability of young neutron stars with high whole-surface temperatures.

2 ELECTROMAGNETIC SHOWER DEVELOPMENT

The later stages in the development of an electromagnetic shower whose primary particle of energy $E_0 \sim 10^2 - 10^3$ GeV moves parallel with $\mathbf{B}$ are determined principally by Coulomb pair production, Compton scattering, and by magnetic bremsstrahlung activated by multiple Coulomb scattering. Formation of the giant-dipole state is by far the most important nuclear interaction and occurs predominantly in the very late stages of shower development, within the photon energy band of $15 - 30$ MeV. Higher-multipole interactions have much smaller cross-sections and appear only at greater energies where the photon track length within the shower is smaller. The principal processes will be described in some detail in Sections 2.1 - 2.4, but in Section 2.4, other processes that are present in showers at $B > B_c$ are discussed briefly.

The natural unit of length for shower development at $B \ll B_c$ is the radiation length defined in terms of the Bethe-Heitler formulae (Bethe 1934) for the bremsstrahlung and pair production cross-sections. This convention is adopted here for convenience, though with cross-sections modified by a screening wavenumber defined by the matter density at the neutron-star surface. For $10 < B_{12} < 10^3$, where B_{12} is the magnetic flux density in units of 10^{12} G, we fit the ion number densities found by Medin & Lai (2006) by the expression

$$N = 2.7 \times 10^{25} Z_{26}^{-0.7} B_{12}^{1.2} \mathrm{cm}^{-3}. \tag{4}$$

The radiation length is then,

$$l_r = 2.4 \times 10^{-2} Z_{26}^{-1.3} B_{12}^{-1.2} \left(\ln \left(62 Z_{26}^{1/2} B_{12}^{-1/2} \right) \right)^{-1} \text{cm} \quad (5)$$

in which the logarithm, whose argument is derived from the infinite linear-chain ion separations given by Medin & Lai, replaces the $\ln(183Z^{-1/3})$ of the Bethe-Heitler formulae. This is extremely small; $l_r = 5.2 \times 10^{-5}$ cm at $B = 10^{14}$ G and $Z = 26$. The Bethe-Heitler mean free path for Coulomb pair production is $9l_r/7$.

The electron or positron dispersion relation in a field of magnetic flux density $\mathbf{B}$ is

$$E = \left(1 + p_\parallel^2 + 2n\frac{B}{B_c} \right)^{1/2}. \quad (6)$$

Unless otherwise stated, all electron or photon energies and momenta will be expressed in units of the electron rest energy and of mc. The classical momentum components $p_\parallel$ and $p_\perp$ are parallel with and perpendicular to $\mathbf{B}$, and n is the Landau quantum number. The conserved quantities in electron-photon interactions are the total energy and the parallel momentum component, but not $p_\perp$ except in the correspondence principle limit of $n \gg 1$. States of $n \sim 1$ are important at $B > B_c$, and are best regarded as of one-dimensional motion, given by $p_\parallel$, with effective mass $\sqrt{1 + 2nB/B_c}$. It is unfortunate that there appear to be no published calculations of the Coulomb bremsstrahlung and pair creation cross sections that are valid for $B \sim B_c$.

2.1 Magnetic bremsstrahlung

Multiple Coulomb scattering of a shower electron or positron is a sequence of ascending (or descending) Landau transitions from its creation state $n_\pm$. For $B > B_c$, the classical ($n \gg 1$) energy loss-rate expression for magnetic bremsstrahlung (see the reviews of Erber 1966 and of Harding & Lai 2006) is not valid, particularly in the later stages of shower development where the dominant process has two stages: Coulomb excitation of the $n = 0 \to 1$ transition followed by cyclotron emission. In order to see that this is so, and in the absence of an accurate expression, we are obliged to adopt the correspondence principle for an order of magnitude estimate and therefore assume that a momentum transfer $q_\perp \sim \sqrt{2B/B_c}$ from the nuclear Coulomb field is necessary for an $n = 0 \to 1$ transition. The mean free path, derived from the Rutherford scattering formula for $q_\perp$ and from the number density of equation (4), is then,

$$\lambda = 5.1 \times 10^{-5} B_{12} \ln \left(62 Z_{26}^{1/2} B_{12}^{-1/2} \right) l_r, \quad (7)$$

equivalent to $\sim 10^2$ Coulomb excitations per radiation length at 10^{14} G. The relativistic quantum-cyclotron emission rate can be approximated by $\tau_{ce}^{-1} \approx 1.0 \times 10^{-3} \omega_B$ in the $p_\parallel = 0$ frame of reference, where ω_B is the classical cyclotron angular frequency. This expression is almost stationary with respect to n and B/B_c in the vicinity of 10^{14} G (see Fig. 3 of Harding & Lai 2006). The condition that the cyclotron emission rate exceeds the Coulomb excitation rate is that,

$$\gamma_0 < \frac{\lambda}{c\tau_{ce}} \approx 0.7 B_{12}^{0.8} Z_{26}^{-1.3}, \quad (8)$$

in which γ_0 is the Lorentz factor for transformation from the $p_\parallel = 0$ frame to that of the rotating star. Thus the statement that $n = 0 \rightleftharpoons n = 1$ transitions are the dominant processes is valid in the very late stages of shower development at 10^{14} G and we can conclude that in this region electrons or positrons lose their energy over distances one or two orders of magnitude smaller than l_r.

From the kinematic conservation laws, the maximum photon transverse momentum (that is, $\perp$ to $\mathbf{B}$) in cyclotron emission for an $n \to n'$ transition is,

$$k_\perp = \sqrt{1 + 2n\frac{B}{B_c}} - \sqrt{1 + 2n'\frac{B}{B_c}}. \quad (9)$$

Because the low-energy part of the electron-positron spectrum is the source of most GDR-band photons, we shall show, in Section 2.3, that this limit on photon transverse momentum is of crucial importance in estimating their total track length in the shower. But earlier in the shower, for values of γ_0 about two orders of magnitude larger such that $\gamma_0 c\tau_{ce} > l_r$, Coulomb bremsstrahlung is the more important energy-loss process.

2.2 Coulomb bremsstrahlung

The cross-section for Coulomb bremsstrahlung at $B > B_c$ must differ from the Bethe-Heitler formula, but no result appears to have been published at the present time. For the emission of a photon of momentum $\mathbf{k}$ at an angle θ with $\mathbf{B}$, by an electron or positron of initial momentum $p_\parallel$ with a change of Landau quantum number $n \to n'$, the required momentum transfer components are, in the limit $p_\parallel \gg 1$,

$$\begin{aligned} q_\parallel &= k(1 - \cos\theta) + (n' - n)\frac{B}{B_c}\frac{1}{p_\parallel} + \\ &\quad \frac{k}{2p_\parallel^2}\left(1 + 2n'\frac{B}{B_c}\right), \\ q_\perp &\sim \sqrt{\frac{2B}{B_c}}, \qquad \text{for} \qquad n' \neq n \end{aligned} \quad (10)$$

For transitions with $n' \neq n$, the threshold longitudinal momentum transfer is larger than its zero-field value of $q_\parallel = k/2p_\parallel^2$ and it is therefore possible that the true cross-section is reduced in comparison with the Bethe-Heitler value. But this merely increases the electron track length in the earlier stages of shower development, with which we are not concerned, and does not affect the photon total track length in the GDR band.

2.3 Coulomb and magnetic pair creation

A photon with momentum $\mathbf{k}$ at an angle θ with $\mathbf{B}$ can convert to an electron-positron pair with Landau quantum numbers $n_\pm$. Only the longitudinal component of momentum transfer from the nuclear Coulomb field appears in the kinematic conservation laws and the threshold is,

$$\begin{aligned} q_\parallel &= k(1 - \cos\theta) - \frac{1}{2p_{\parallel+}}\left(1 + 2n_+\frac{B}{B_c}\right) - \\ &\quad \frac{1}{2p_{\parallel-}}\left(1 + 2n_-\frac{B}{B_c}\right), \end{aligned} \quad (11)$$

which is valid in the limit $p_{\parallel\pm} \gg 1$ and has two regions of interest; $q_\parallel = 0$ and $q_\parallel \neq 0$. It is again unfortunate that there appears to be no published cross-section for the

Coulomb process at finite $q_{\|}$, and in finding the total photon track length in this case, we shall assume the Bethe-Heitler mean free path, multiplied by an unknown correction factor $\eta_p > 1$. However, $q_{\|} = 0$ is possible at a finite θ, and the associated transverse momentum component $k_{\perp}$ is the threshold for single-photon magnetic pair production. The individual thresholds that are important here are,

$$\begin{aligned} k_{\perp} &= 2, & n_{+} = n_{-} = 0, \\ k_{\perp} &= 1 + \sqrt{1 + 2\frac{B}{B_c}}, & n_{\pm} = 0, \quad n_{\mp} = 1. \end{aligned} \tag{12}$$

Transitions to the Landau state $n_{-} = n_{+} = 0$ are not allowed for photons polarized so that the electric vector is parallel with $\mathbf{k}\times\mathbf{B}$ (Semionova & Leahy 2001, see also Harding & Lai 2006), but are allowed for photons polarized with electric vector perpendicular to $\mathbf{k}\times\mathbf{B}$. The classical expression for the conversion rate, which is valid in the $n_{\pm} \gg 1$ limit, is a function of $k_{\perp}B/B_c$ (see Erber 1966; Harding & Lai 2006), and is not applicable here at fields $B > B_c$ with $n_{\pm} \sim 1$. But the rate $R_{\gamma ee} = 1.2 \times 10^6$ cm^{-1} that it predicts for 30 MeV photons at 10^{14} G, immediately above the higher of the two thresholds given by equations (12), is so large that, whatever its inadequacies, it must be presumed that magnetic conversion occurs within a very small fraction of a radiation length. Thus photons with $k_{\perp}$ above the higher threshold do not contribute significantly to the total photon track length or to the photoproduction of protons through GDR formation. Photons in the allowed polarization state can convert above the lower threshold but the rate is reduced by a factor of the order of $(k_{\perp}/k)^2 \approx 10^{-3}$ and is approximately an order of magnitude smaller than that for Coulomb conversion. A further small contribution to the magnetic conversion rate must exist because the correct electron and positron propagators are those for passage through condensed matter, not free space. Therefore, Coulomb interaction introduces a small component of $n_{\pm} = 0$, $n_{\mp} = 1$ into the $n_{+} = n_{-} = 0$ state. But this is relatively unimportant and has not been studied in detail here.

Comparison of equations (9) and (12) shows that all photons emitted by cyclotron decay of Landau states n to the ground state $n = 0$ do not exceed the thresholds for subsequent single-photon magnetic pair production provided,

$$\begin{aligned} \sqrt{1 + 2n\frac{B}{B_c}} &< 3, \\ \sqrt{1 + 2n\frac{B}{B_c}} &< 2 + \sqrt{1 + \frac{2B}{B_c}}, \end{aligned} \tag{13}$$

where the two inequalities refer to the two thresholds of equation (12). The angular distribution of the cyclotron decay photon in the $p_{\|} = 0$ frame of reference considered in Section 2.1 is approximately isotropic (see Fig. 3 of Latal 1986), so that because equation (9) gives the maximum possible value of $k_{\perp}$, photons with $k_{\perp}$ below the maximum may well fall below the thresholds (12) even if the above inequalities are not satisfied. It is interesting to note the first inequality fails for the minimum Landau quantum number $n = 1$ for all $B > 4B_c$ whilst the second is satisfied for all B.

2.4 Compton scattering and other processes

Compton scattering is the most important of the processes of second order in the fine structure constant that are not considered in previous Sections. It can significantly modify the very late stages of shower development at $B > B_c$. Extensive cross-section calculations have been performed by Gonthier et al (2000) for a wide interval of photon energy. For the purposes of estimating its significance, the mean free path equivalent to the Thomson cross-section σ_T for the electron density given by equation (4) with $B = 10^{14}$ G and $Z = 26$ is $\lambda_T = 8.5 \times 10^{-6}$ cm, which is less than a radiation length. The total cross-sections (see Fig. 2 of Gonthier et al) are negligible at photon energies $k > 10^3$, but are significant in the GDR band. Scattered photons with $k_{\perp}$ exceeding the higher threshold (12) undergo prompt magnetic conversion. Gonthier et al have shown that the exact cross-section in this region is well approximated by the non-magnetic Klein-Nishina formula, a very convenient result which enables us to establish that scattered photons with $k_{\perp}$ below the magnetic pair thresholds comprise only a very small, and for present purposes, negligible fraction of the total Compton cross section. Thus the cross-sections that effectively convert photons to electron-positron pairs can be obtained directly from Fig. 2 of Gonthier et al. They are slowly varying functions of B/B_c. We shall assume a photon energy of 21 MeV for GDR formation in nuclei of atomic number $10 \leqslant Z \leqslant 26$ (see the review of Hayward 1963). At this energy, the Compton cross sections are $\sigma_C = 3.6 \times 10^{-2}\sigma_T$ for $B = B_c$ and $4.8 \times 10^{-2}\sigma_T$ for $B = 10B_c$. At higher energies, the cross section decreases, being approximately $\propto k^{-1}$.

The other second-order processes which should be considered briefly involve the blackbody radiation field, whose photons are typically of energy $k_{bb} \sim 10^{-3}$. Inverse Compton scattering can give photons exceeding the pair production thresholds (12), but only for electron momenta greater than $k \sim 10^4$. Similarly, two-photon pair production is possible, but with a threshold $k = k_{bb}^{-1}$ and a negligible transition rate because the blackbody photon number density is many orders of magnitude smaller than the ion densities given by equation (4). Direct production of electron-positron pairs (the trident process) is of third order and has no significant effect on shower development. Photon splitting, also of third order, has a small transition rate, saturating at $R_{ps} \approx 0.4k^{-1}k_{\perp}^6$ cm^{-1} at $B > B_c$ (see the review of Harding & Lai 2006).

The remaining process that does change shower development is the Landau-Pomeranchuk-Migdal effect for which we refer to the extensive review given by Klein (1999), also to the work of Hansen et al (2004) for its laboratory verification at electron energies of several hundred GeV. In condensed matter at zero field, multiple Coulomb scattering removes the coherence that the Bethe-Heitler formulae assume over distances of the order of $q_{\|}^{-1}$, where $q_{\|}$ is the very small longitudinal momentum transfer from the nucleus, and so reduces the cross sections, particularly for bremsstrahlung production of low-energy photons. This is almost certain to be a feature of the showers considered here owing to the high density of matter at the neutron-star surface. But it is not possible to make even qualitative estimates of the effect because the nature of multiple Coulomb scattering is changed at $B > B_c$ (see Section 2.1) and there is also a strong one-

dimensional ordering of ions parallel with **B** arising from the formation of bound moleculsr chains (see Medin & Lai 2006) which will be of significance for questions of coherence of the Coulomb bremsstrahlung and pair creation amplitudes. It is almost certain that in the early stage of shower development considered here, the LPM effect leads to an inward displacement of the GDR photon band region, though without significant change in its total track length. We shall see in Section 3 that the formation of protons is also not significantly changed but that their time-scale for diffusion to the surface is modified in a non-trivial way as described in Section 4.3.

2.5 Photon track length in a shower

The distribution of photon track length is found from electromagnetic shower theory. On dimensional grounds, the track length per unit interval of photon energy k such that $E_c \ll k \ll E_0$ in a shower of primary electron energy E_0 must be of the form,

$$G(k) = \frac{yE_0 l_r}{k^2} \tag{14}$$

in which y is a dimensionless parameter and E_c is the critical energy at which electron or positron ionization and bremsstrahlung energy-loss rates are equal. In the zero-field case, where there is more certainty in the fundamental processes of shower development, it is possible to obtain reliable values of y. But at $B > B_c$, we are obliged to adopt a more elementary and analytical approach, usually referred to as Approximation A, which includes only the most significant processes which are bremsstrahlung and pair creation (Landau & Rumer 1938). But even at this level, there are difficulties because the relative importance of Coulomb and magnetic bremsstrahlung changes as the shower develops. The later stages of shower development with which we are concerned here are almost completely determined by magnetic bremsstrahlung, as described in Section 2.1. Therefore, a modified Approximation A, in which throughout the development of the shower, Coulomb bremsstrahlung is replaced by magnetic bremsstrahlung, has been used to calculate an approximate value of y. Pair creation is assumed to be determined by the Bethe-Heitler formula with the unknown correction factor η_p retained (Section 2.3). To obtain the total photon track length, only the form of the photon spectrum is needed and initially, we assume the number density $\propto k^{-2/3}$ per unit interval of k which is valid in the classical magnetic bremsstrahlung limit of $n \gg 1$. Following closely the procedures of Landau & Rumer (1938) and Nordheim & Hebb (1939), we find that $G(k)$ is given by the inverse Mellin transform,

$$G(k) = \frac{-1}{2\pi i}\int_{-i\infty+\zeta}^{i\infty+\zeta} \frac{\eta_p l_r E_0^s}{k^{s+1}D(s)} ds, \tag{15}$$

in which

$$D(s) = \left(\frac{2}{s+1} - \frac{8}{3}\frac{1}{s+2} + \frac{8}{3}\frac{1}{s+3}\right) + \frac{7}{9}\left(s+\frac{1}{3}\right)\int_0^1 \frac{\xi^s - 1}{(1-\xi)^{2/3}} d\xi \tag{16}$$

with ζ chosen so that all poles lie to the left of the contour. To obtain the term linear in E_0 we retain only the contribution from the $s = 1$ pole and find $y = 0.608\eta_p$. Apart from the unknown factor, this differs little from the zero-field Approximation A in which, for an approximate photon number density $\propto k^{-1}$, the value is $y = 0.47$.

But as shown in Section 2.1, the magnetic bremsstrahlung classical limit is not reached in the later stages of shower development. Instead, the dominant processes are Landau transitions $n = 0 \rightleftharpoons n = 1$ in which the angular distribution of the cyclotron decay photons in the $p_{\parallel} = 0$ frame of reference is approximately isotropic. Thus a Lorentz transformation to the frame of the rotating star produces a photon number density distribution which is a constant per unit interval of k. In this case, the denominator function $D(s)$ in equation (15) is replaced by

$$D_c(s) = \frac{2}{s+1} - \frac{8}{3}\frac{1}{s+2} + \frac{8}{3}\frac{1}{s+3} - \frac{7}{9}s, \tag{17}$$

giving the result $y = 0.871\eta_p$. It is worth noting that the Approximation A results for total photon track length are independent of the absolute rates of bremsstrahlung emission. Even the effects of quite substantial changes in its spectral shape are relatively small, being limited to small increases in y for the harder spectra.

Two processes not included in Approximation A, Compton scattering and energy loss by ionization, can affect the late stages of shower development. The principal effect of Compton scattering is a reduced effective mean free path for pair creation by photons in the GDR band and therefore, a decreased total photon track length available for GDR formation. This is important and can be allowed for by the replacement,

$$\eta_p \to \frac{\eta_p}{1+\eta_p x_C}, \qquad x_C = \frac{9}{7}Nl_r Z\sigma_C, \tag{18}$$

in equation (15), with $\sigma_C(k)$ evaluated at the GDR formation energy $k_{gd} = 41$.

It is known through comparison of Approximations A and B that in the zero-field case, electron and positron ionization energy loss produces some reduction in y because GDR-band energies are of the same order of magnitude as the critical energy E_c. But the total electron-positron track length in the later stages of shower development at $B > B_c$ is much reduced because energy loss by magnetic bremsstrahlung predominates and occurs over lengths one or two orders of magnitude smaller than l_r (see Section 2.1). Thus the relative importance of ionization energy loss is much reduced near the GDR band and we attempt no correction for it.

On this basis, we shall adopt the track-length estimate given by equation (14) in which yl_r is found from equations (4), (5), (15), (17) and (18), with the parameter η_p as an unknown.

3 PROTON PRODUCTION

The mean number of protons formed in a shower per unit interval of primary electron energy is

$$W_p = \langle NI_{\gamma p}\rangle G(k_{gd})/E_0, \tag{19}$$

in which $I_{\gamma p}$ is the energy-integrated cross section for giant dipole resonance formation and decay into channels containing one or more protons, and the square brackets denote an

average over the depth distribution of GDR-band photons in the shower. This latter function is unknown in detail, owing to the gradual transition from Coulomb to magnetic bremsstrahlung as the dominant process in shower development and to the existence both of the unknown parameter η_p and the LPM effect, but it is certainly very small near the surface and increases to a maximum at a depth which is likely to be of the order of $10l_r$. Therefore, we are obliged to evaluate the ion number density N and $I_{\gamma p}$ at a mean atomic number Z. The integrated cross-section is expressed as the electric-dipole sum-rule value (Hayward 1963) multiplied by a factor x_p,

$$I_{\gamma p} = 117.4\,(x_p Z)\left(1 - \frac{Z}{A}\right) \quad mc^2 \quad \mathrm{mb} \tag{20}$$

where A is the nuclear mass number. Measured cross-sections for light elements $10 \leqslant Z \leqslant 26$ provide evidence for a value $x_p \approx 0.5$ (Hayward 1963; Kerkhove et al 1985; Assafiri & Thompson 1986). But in the circumstances of shower development below the neutron-star surface, the problem of estimating the effective value of x_p is more complicated.

The protons formed in GDR decay lose energy by ionization, are quickly thermalized, and diffuse to the surface with negligible probability of secondary nuclear interaction. However, this is not so in the case of GDR-decay neutrons which lose energy and diffuse to the surface by a sequence of nuclear elastic scatterings in the course of which capture in a transition $A \rightarrow A+1$ may occur with a significant probability which is, however, hard to estimate. Consequently, nuclei in the shower region may be neutron-rich. There is some evidence (Fultz et al 1974; O'Keefe et al 1987) that x_p decreases with increasing nuclear asymmetry $A-2Z$, possibly owing to the associated divergence between neutron and proton separation energies. A simple model of the effect of this increase in nuclear asymmetry is as follows. The energy-integrated cross-section for neutrons is given by equation (20) with the substitution $p \rightarrow n$ and $x_p + x_n = a \approx 1$. Suppose that neutrons diffuse to the surface with survival probability ϵ_n and that the captured fraction $1 - \epsilon_n$ increases nuclear asymmetry. This produces a bias toward GDR decay by neutron emission. To take account of this, let $x_n = Hx_p$, in which H is a function of $A-2Z$. The steady-state condition for equal neutron and proton loss rates is then $x_p = \epsilon_n x_n = H\epsilon_n x_p$, from which we find that ϵ_n determines the asymmetry function H and $x_p = a\epsilon_n(1+\epsilon_n)^{-1}$. In the limit of small ϵ_n, the nuclear asymmetry is, of course, limited by β-instability, the relevant time-scale being that in which the corotational current density $\rho_0 c$ removes the equivalent of one radiation length of matter, $t_{rl} = Nl_r AeP/B \sim 10^2$ s for $B = 10^{14}$ G and rotation period $P = 1$ s.

Values of W_p found from equation (19) are given in the final column of Table 1 for intervals of Z and of B that might permit the $\mathbf{E}_\parallel \neq 0$ boundary condition at the polar cap. They assume the parameters $\eta_p = 1$ and $x_p = 0.5$. To allow for the effect of neutron capture, the numbers in the right-hand column should be multiplied by $2a\epsilon_n(1+\epsilon_n)^{-1}$. Multiplication by $\eta_p(1+x_C)(1+\eta_p x_C)^{-1}$ allows for values of $\eta_p > 1$. In the latter case, it is worth noting that even in the limit $\eta_p \gg 1$, the existence of the Compton effect, which has been extensively studied at $B > B_c$, maintains shower development. Proton formation rates in the Table do not vary greatly as functions of Z or of B and are extremely high, so that a primary electron of 10^3 GeV creates between 200 and 500 protons. It is, of course, the case that many approximations have been made in Sections 2.1 - 2.4 but these do not have a great effect on the late stages of shower development in which most of the total track length given by equation (14) is generated. It is principally for this reason that we believe Table 1 provides a reliable estimate of the true rates.

Table 1. The number of protons created per unit primary electron energy, in units of mc^2, are given in the right-hand column for values of the mean nuclear charge Z in the region of the shower maximum and of the magnetic flux density B in units of the critical field $B_c = 4.41 \times 10^{14}$ G. Also given are the radiation length l_r in neutron-star surface matter and the parameter x_C that allows for the Compton effect correction given by equation (18). It is assumed here that the correction factor applied to the Bethe-Heitler pair creation mean free path is $\eta_p = 1$ and that the strength of the giant dipole-resonance proton decay channels is given by $x_p = 0.5$. We refer to Section 3 for the correction to be applied given different values of these two parameters.

Z	BB_c^{-1}	l_r cm	x_c	W_p $(mc^2)^{-1}$
10	1	5.0×10^{-4}	0.76	2.1×10^{-4}
10	3	2.0×10^{-4}	1.30	2.4×10^{-4}
10	10	9.2×10^{-5}	2.96	2.8×10^{-4}
18	1	2.0×10^{-4}	0.37	1.5×10^{-4}
18	3	7.3×10^{-5}	0.58	1.7×10^{-4}
18	10	2.9×10^{-5}	1.11	2.2×10^{-4}
26	1	1.1×10^{-4}	0.23	1.0×10^{-4}
26	3	4.0×10^{-5}	0.36	1.2×10^{-4}
26	10	1.5×10^{-5}	0.64	1.5×10^{-4}

3.1 Proton diffusion

Diffusion of protons to the surface is very rapid. The distribution of proton formation depth depends on a number of factors and is not easily obtained from shower calculations such as those described in Section 2.5. The dominant process of photon creation changes during shower development from Coulomb to magnetic bremsstrahlung (see Sections 2.1 and 2.2). The LPM effect, considerd in Section 2.4, displaces shower development inward as does the unknown factor $\eta_p > 1$, particularly the region of low-energy photon formation which is relatively compact. On this basis, it is convenient to assume a normalized proton formation depth distribution given by

$$g_p(z) = \delta(z - z_p), \tag{21}$$

though with the reservation that z_p is not well known. Let the time interval between creation and arrival at the surface be $t - t' = \tau t_p$ with τ dimensionless. For a semi-infinite homogeneous medium, the probability per unit interval of τ that a proton created at depth $z < 0$ arrives at the surface $z = 0$ is,

$$P(\tau, z) = -z\left(\frac{2}{3}\pi\tau^3 z_p^2\right)^{-1/2} \exp\left(-\frac{3z^2}{2z_p^2\tau}\right), \tag{22}$$

in which $z_p^2 = 3Dt_p$ and D is the diffusion coefficient. Thus the probability of arrival at the surface with delay τ is,

$$f_p(\tau) = \int_{-\infty}^{0} dz g_p(z) P(\tau, z) = \left(\frac{2}{3}\pi\tau^3\right)^{-1/2} \exp\left(-\frac{3}{2\tau}\right). \quad (23)$$

The solid at the polar-cap surface consists of linear molecular chains of ions, parallel with $\mathbf{B}$ and normal to the surface, probably ordered so as to form a three-dimensional lattice. An approximate upper limit for D can be obtained from the proton thermal kinetic energy and from the ion separation $a_s \approx 5 \times 10^{-9} Z_{26}^{1/2} B_{12}^{-1/2}$ cm found from the calculations of Medin & Lai (2006). It is,

$$D = a_s \sqrt{\frac{1}{\beta m_p}} \approx 4 \times 10^{-2} \sqrt{\frac{T_6 Z_{26}}{B_{12}}} \quad \mathrm{cm^2 s^{-1}}, \quad (24)$$

where $\beta^{-1} = k_B T$ and T_6 is the polar-cap surface temperature in units of 10^6 K. This is based on the reasonable assumption that, within the solid structure, the proton potential barriers are at most $\sim k_B T$. For $z_p = 10 l_r$ and $T_6 = 1$, the characteristic time for diffusion to the surface is then $t_p \approx 10^{-5}$ with the values of l_r given in Table 1.

The condition of the surface at such temperatures is that it remains solid. The melting temperatures predicted by the one-component Coulomb-lattice condition given by Slattery, Doolen & DeWitt (1980) are close to 10^7 K at $B = B_c$ and are consistent with the melting temperature shown in Fig. 1 of the paper by Potekhin et al (2003). (We shall see in Section 4 that the longitudinal thermal conductivity is so large that the same statement can be made with confidence of matter at the shower maximum $z = z_p$.) Protons reaching the surface are accelerated parallel with the magnetic flux density $\mathbf{B}$ either immediately or, if their flux exceeds the corotation value, from the top of the gravitationally-bound proton atmosphere that is then formed.

It is also worth noting that the evaporation of ions at the surface is determined purely by the thermal excitation of lattice degrees of freedom. Direct interaction with the inward flux of photons produced by curvature radiation from reverse electrons is not significant in this respect. Photodisintegration of nuclei within a small number of lattice planes immediately below the surface has negligible probability. Photoelectric or bound-free cross-sections for the removal of electrons from ions are large, but the momentum transfer to the nucleus is determined by the Fourier transform of the bound electron wave function and its maximum value is therefore of the order of $137\hbar a_0^{-1}\sqrt{B/B_c}$, where a_0 is the Bohr radius. The equivalent recoil energy is $140 A^{-1} B B_c^{-1}$ eV for an ion of mass number A, which is negligible.

The magnitude of the photoelectric cross section is itself a source of reverse electron flux (see Jones 1981). Electrons removed from partially-ionized atoms accelerated through the polar-cap blackbody radiation field form a significant flux at the polar-cap surface. Given the large values of W_p, a direct consequence is that the time-averages of the electron-positron, ion and proton current densities J_e, J_Z and J_p on open magnetic flux lines satisfy $J_p \gg J_{e,Z}$. The time-averaged particle flux above the polar cap consists almost entirely of protons.

3.2 The structure of the proton and ion atmosphere

Ion cohesive energies are by no means negligible at $B \approx B_c$. Those calculated by Medin & Lai (2006) for $Z = 26$ can be expressed approximately as $E_c = 0.16 B_{12}^{0.7}$ keV in the interval $10 < B_{12} < 100$ which is of interest here, and at the lower end of this interval are in fair agreement with the cohesive energy of 0.92 keV for $Z = 26$ at $B_{12} = 10$ calculated by Jones (1985) using a different model for the three-dimensional lattice. Thus $\beta E_c \gg 1$ and, in consequence, there is a large density discontinuity at the transition between the solid phase and the very thin gravitationally-bound atmosphere. At lower magnetic fields, such that $\beta E_c \sim 1$, the discontinuity is less obvious and the atmosphere more dense.

The part of this atmosphere that is in local thermodynamic equilibrium exists within $0 < z < z_1$, where we assume that z_1 is the surface of last scattering, roughly defined here as the altitude beyond which number densities are so small that scattering of electrons and ions can be neglected. The proton number density in this interval is in general many orders of magnitude smaller than those of ions and electrons which therefore determine the atmospheric structure. Let us assume for the purposes of illustration that there is present only one ion species and that it is partially ionized with charge $\tilde{Z}$ and mass number A. The standard relations between number densities and chemical potentials for free Boltzmann particles (allowing for non-zero B in the case of electrons), with the constraint of electrical neutrality (strictly charge density ρ_0), require that in local thermodynamic equilibrium there must be a very small electrostatic potential $\Phi(z)$ given by

$$e\Phi(z) = -\frac{1}{\tilde{Z}+1}\left(M_{A,\tilde{Z}} - m\right) gz, \quad (25)$$

where g is the local gravitational acceleration. This result is not much changed if allowance is made for the presence of different ionization states (see Jones 1986). The scale height of the atmosphere is very small and, in the general case, the structure is of successive layers of increasing $\tilde{Z}/A$ as the altitude z increases. Equation (25) can be used to obtain the scale height of the very small proton component. It is negative, showing that protons cannot be in thermodynamic equilibrium at $z < z_1$ and are accelerated across the surface of last scattering.

The proton number density equivalent to the corotational charge density is given by $N_p e v_p / c = \rho_0$ and, even for polar-cap thermal values of velocity v_p, is much smaller than ion densities at z_1. Thus as particles move out in Landau states parallel with $\mathbf{B}$ to $z > z_1$, the condition of electrical neutrality again requires an electrostatic potential that is independent of the small proton number and is also given precisely by equation (25). The proton potential energy at $z > z_1$ is

$$m_p(z - z_1)g + e\Phi(z) - e\Phi(z_1) = m_p(z - z_1)g - \frac{1}{\tilde{Z}+1}\left(M_{A,\tilde{Z}} - m\right)(z - z_1)g, \quad (26)$$

which is evidently negative. The relatively small numbers of protons are again accelerated whilst ions, with smaller charge-to-mass ratios, are retained by the gravitational field. Thus the protons are preferentially selected for the process

of inertial acceleration that follows. If the flux of protons at $z = 0$ is greater than is needed for the corotational current density, the surplus forms an atmosphere at $z > z_1$. This ordering has significant consequences for plasma formation, as will be seen in Section 4.

3.3 Reverse electron flux from photoelectric ionization

Inertial acceleration under the $\mathbf{E}_{\parallel} = 0$ polar-cap surface boundary condition was first described by Michel (1974). Owing to current conservation, an excess of positive charge is present immediately above the surface in the non-relativistic stage of ion acceleration. This is the source of an electric field,

$$E_i \approx \left(\frac{8\pi\rho_0 Mc^2}{\tilde{Z}e}\right)^{1/2} = 5.0 \times 10^4 \left(\frac{-AB_{12}\cos\psi}{P\tilde{Z}}\right)^{1/2} \quad (27)$$

expressed in electrostatic units (esu) at $z > z_i$, which is the roughly defined altitude at which ion motion becomes fully relativistic,

$$z_i \approx \left(\frac{Mc^2}{18\pi\rho_0 \tilde{Z}e}\right)^{1/2} = 4.1 \times 10^1 \left(\frac{-AP}{B_{12}\tilde{Z}\cos\psi}\right)^{1/2} \text{cm} \quad (28)$$

where P is the rotation period and ψ is the angle between $\mathbf{\Omega}$ and $\mathbf{B}$. (In obtaining these approximate results, the ion charge $\tilde{Z}$ has been assumed constant in the interval $0 < z < z_i$.)

The change of ionization from partial ($\tilde{Z}$) to complete (Z) is the result of photoelectric absorption of blackbody photons whose energies have been boosted by Lorentz transformation to the rest frame of the accelerated ion. A simple model of the process has been described previously in which transitions occur sequentially in order of increasing electron separation energy at altitudes for which unit optical depth of blackbody radiation field for a particular transition is reached. In this way, a compact approximate expression for the mean reverse electron energy at the polar-cap surface per accelerated unit positive charge can be obtained (Jones 1981; see equations 15 - 22, and Table 2 of that paper). The prefactor in the expression for the electron separation energies assumed there was obtained for smaller magnetic fields ($B_{12} \sim 1$) than those considered here, but comparison with the recent calculations of Medin & Lai (2006) for multiply ionized C and Fe ions shows that separation energies at $B > B_c$ can found with sufficient accuracy for present purposes by scaling from that expression. Following this procedure, the mean electron energy per accelerated unit nuclear charge is estimated to be

$$\left(\frac{Z - \tilde{Z}}{Z}\right)\bar{\epsilon}_e \approx 5.6 \times 10^4 Z_{26}^{0.85}(0) B_{12}^{0.5} T_6^{-1.0} \quad mc^2, \quad (29)$$

in which $Z(0)$ is the mean nuclear charge at the surface $z = 0$. These values assume the acceleration field is uniform at altitudes small compared with the polar-cap radius, but are quite insensitive to variations in its strength because photoionization occurs quite promptly when the blackbody radiation field is boosted, in the rest frame of the accelerated ion, to the electron separation energy. The altitude at which this occurs has little effect on the reverse-electron energy flux. We assume a value $E_{\parallel} = E_i = 10^6$ esu. Equation (29) obviously fails for very light ions, $Z < 4 - 5$, which are likely to be completely ionized in the LTE region at $z < z_1$. It can be combined with an approximate parameterization of the values for W_p given here in Table 1,

$$W_p = 7.0 \times 10^{-5} \langle Z_{26}^{-0.76} \rangle_{sm} B_{12}^{0.12} \quad (mc^2)^{-1}, \quad (30)$$

where square brackets denote the average nuclear charge in the region of the shower maximum, to obtain an expression for the number of protons produced per unit positive nuclear charge accelerated. It is,

$$K = K_0 Z^{0.85}(0) \int g_p(z) dz$$

in which,

$$K_0 = 0.24 \langle Z_{26}^{-0.76} \rangle_{sm} B_{12}^{0.62} T_6^{-1.0}. \quad (31)$$

We note again that this is invalid for very light ions, $Z < 4 - 5$. It is also not valid at much smaller fields where K tends to an asymptotic value. Here, W_p is given by the zero-field value of the shower parameter y (see Section 2.5) and by placing $B_{12} = 1$, so that equation (30) is replaced by $W_p = 3.8 \times 10^{-5} \langle Z_{26}^{-0.76} \rangle_{sm} (mc^2)^{-1}$. Electronic separation energies concerned here approach asymptotically their zero-field values at $B_{12} < 1$ so that equation (29) should be evaluated at $B_{12} = 1$ to obtain an estimate of the asymptotic value of K_0,

$$K_0 = K_{0as} = 0.13 \langle Z_{26}^{-0.76} \rangle_{sm} T_6^{-1.0}. \quad (32)$$

Equations (31) and (32) are an under-estimate of K owing to our neglect here of a further source of reverse electrons that is difficult to quantify, specifically pair creation following inverse Compton scattering of blackbody photons by the photoelectrons or the consequent conversion of curvature radiation photons at higher altitudes. The electron and photon momenta are so oriented that the necessary electron Lorentz factor, $10^4 - 10^5$, is easily reached. For the significance of the outward accelerated positrons as a source of pairs, we refer to Harding & Muslimov (2002).

4 APPLICATION TO POLAR CAPS

Neutron stars with spin direction such that $\mathbf{\Omega} \cdot \mathbf{B} < 0$ and polar-cap surface boundary condition $\mathbf{E}_{\parallel} = 0$ must presumably exist but the extent to which they are present in the observed population of radio pulsars is an unsolved problem. The physical existence of case (iii) of Section 1, the boundary condition $\mathbf{E}_{\parallel} \neq 0$, is less certain in the observed neutron star population and is a question to which proton production in polar-cap reverse electron flow is very relevant. This section attempts to estimate the interval of polar-cap magnetic field that would support this condition and to investigate the nature of the plasma formed. It does not attempt to consider the actual radio emission process.

The distribution of the inferred polar-cap dipole field strengths for the observed radio pulsar population contained in the ATNF Catalogue (Manchester et al 2005) indicates that a large fraction are likely to be too small to support case (iii). But it is interesting to note, though with reservations about the possibility of unknown selection bias, that the nulling radio pulsars listed in Tables 1 and 2 of the paper by Wang, Manchester & Johnston (2007) have a distri-

bution significantly displaced toward larger fields in comparison with the complete ATNF catalogue population. The distribution of $\log_{10} B$, where B is here the listed polar-cap field, for the whole population of 1486 radio pulsars is slightly skewed toward low values but has a mean of 11.98 and a median of 12.06. With neglect of the low-B tail, the distribution has a standard deviation of approximately 0.50. The 63 pulsars for which Wang et al give a finite nulling fraction have a mean of 12.41 and a median of 12.45. The expected standard deviation of the mean for a randomly selected set of 63 pulsars is approximately 0.07 which shows that the set of nulling pulsars have field strengths that are significantly larger than the general population.

We shall see that if the polar-cap fields were identical with their inferred dipole values, few if any pulsars would be able to support the boundary condition $\mathbf{E}_{\parallel} \neq 0$, given the ion cohesive energies referred to at the end of Section 3.2. But in this argument, it is implicit that the dipole origin is positioned at the centre of the star. Outward displacement of the origin along the dipole axis toward the surface of the star would produce considerable amplification of the field at one polar cap and diminution at the other without changing significantly the dipole field at the light cylinder. Beyond this simple case there are, of course, many other possible configurations with enhanced polar-cap fields.

The proton formation rates given in Table 1 are high and any atmosphere formed under boundary condition (ii) is fractionated in the manner described in Section 3.2. Thus the protons undergo inertial acceleration first and, if exhausted, are followed by ions in order of decreasing $\tilde{Z}/A$ if they are present in the atmosphere. Because the time-averaged total open magnetic flux-line current density consists almost entirely of protons and must satisfy the condition $J < \rho_0 c$ it follows that the values of W_p lead directly to an upper limit for the reverse electron-photon energy flux at the polar-cap surface. The temperature limit derived from this for a typical value $W_p mc^2 = 2 \times 10^{-4}$ is,

$$T_{max} = 6.2 \times 10^5 \left(\frac{-B_{12} \cos\psi}{P} \right)^{1/4} \quad \mathrm{K}. \qquad (33)$$

Because it has been obtained from the time-averaged energy flux, it can in principle be exceeded in a time-variable state. But in a time-independent state it can be compared with the critical temperature T_c above which the ion thermal emission rate is high enough to maintain the case (ii) boundary condition $\mathbf{E}_{\parallel} = 0$. In terms of the work function, this is approximately $k_B T_c = 0.030 E_c$. The values of Medin & Lai (2006) for $Z = 26$ and $10 < B_{12} < 100$ can be expressed as $E_c = 0.16 B_{12}^{0.7}$ keV and give $T_c = 5.6 \times 10^4 B_{12}^{0.7}$ K. This can be compared with T_{max} or with T_{res}, the polar-cap blackbody temperature in the absence of reverse-electron flux. (This is approximately the whole-surface blackbody temperature corrected to the local proper frame.) We can see that, regardless of the reverse electron flux, the $\mathbf{E}_{\parallel} \neq 0$ boundary condition can be satisfied only in very old pulsars or in those where the polar cap field is of the order of B_c.

4.1 Medium time-scale variability

The observed field-strength distribution for nulling pulsars indicates that it may be worth considering whether or not the variability time-scales associated with nulling can be a feature of the boundary conditions assumed in case (iii). But first, in Section 4.1, we shall consider case (ii). As we have seen in Section 3, photodisintegration reduces nuclear mass numbers in the region of the shower maximum. Protons diffuse to the surface with negligible probability of nuclear interaction. A fraction ϵ_n of neutrons also escape nuclear capture and reach the surface. Thus nuclei fixed in a lattice plane move gradually toward the surface with a small velocity v. (Our reference to a lattice is merely a convenience. There will be order parallel with $\mathbf{B}$, but also a considerable density of point defects produced by neutron scattering.) Instability against the growth of curvature radiation pairs requires a minimum $E_{\parallel}$ and hence, broadly, that the current density J should not exceed a critical value J_{crit}. We consider first case (ii) with current density $J > J_{crit}$ so that there is no instability against pair formation by curvature radiation. Solution of equation (1), with time-independent boundary conditions and inclusion of inertial acceleration, determines J which is constant. But intuition suggests that the distribution of nuclear charge with depth is not necessarily time-independent because the surface charge $Z(0,t)$ is determined by the surface state at a previous time $t - t_{sm}$, where t_{sm} is the time interval in which nuclei move from the shower maximum to the surface. Let us assume, for simplicity, that $\epsilon_n = 1$.

It will be convenient in this Section to describe the motion of nuclei toward the surface in terms of a Lagrangian depth $\tilde{z}$ which moves with velocity $v(z,t)$ relative to z so that $\tilde{z} = z$ at $t = 0$. The rate of change of nuclear charge within an element at this depth containing a fixed number of nuclei, $N\delta\tilde{z}$ where N is given by equation (4), is

$$N\delta\tilde{z}\frac{\partial Z}{\partial t'} = -g_p(z)\delta z K_0 N(0,t') Z^{1+\nu}(0,t') v(0,t'), \qquad (34)$$

where $\nu = 0.85$ from equations (31) and (32). But $N \propto Z^{\alpha}$ so that the evolution of Z satisfies,

$$Z^{\alpha+1}(-\infty) - Z^{\alpha+1}(0,t) =$$
$$(\alpha+1)\int_{-\infty}^{t} dt' \delta(t - t' - t_{sm}) K_0 Z^{\nu+\alpha+1}(0,t'). \qquad (35)$$

This expression contains the approximation $g_p = \delta(z - z_p) \approx v^{-1}(0,t')\delta(t - t' - t_{sm})$ which is satisfactory provided g_p is a sharply-peaked function displaced below the surface as is expected owing to the nature of shower development and to the LPM effect. It is also unimportant that t_{sm} is to some extent a function of Z-values in the interval $z_p < z < 0$. The initial value of Z is $Z(-\infty)$ and it is assumed constant. A time-independent solution of equation (35) exists, given by,

$$\left(\frac{Z(0)}{Z(-\infty)} \right)^{1+\alpha} = \frac{1}{1 + (1+\alpha)K}$$

But assuming $Z(-\infty) = 26$, and a time-independent charge at the surface so small, $Z(0) = 5$, that equations (29) and (31) begin to fail because few accelerated ions have any bound electrons, we find a self-consistent value $K = 5.5$ for $\alpha = -0.7$. However, this state is not stable because for a charge fluctuation $\delta Z(0,t)$, equation (35) becomes a homogeneous Volterra equation of the second kind with no non-zero square-integrable solution. The charge fluctuation satisfies,

$$\delta Z(0,t) = -(\nu + \alpha + 1) K_0 Z^{\nu}(0, t - t_{sm}) \delta Z(0, t - t_{sm}), \qquad (36)$$

and for the above value of K, grows indefinitely with alternating sign. Ultimately, equation (36) fails and the system alternates between intervals of high nuclear charge at the surface and intervals in which the surface nuclear charge is so low that the reverse electron flux is negligible. The basic unit of time in which t_{sm} is measured is the time t_{rl}, defined in Section 3, for the removal of one radiation length of matter at the corotational current density. This is certainly within the orders of magnitude associated with nulling intervals. The behaviour in time of a complete polar cap is, of course, much more complex because the interval of depth z_p considered here is extremely small compared with the cap radius u_0, and the question arises of the extent (if any) of correlation between different areas of the cap.

It is obvious that there must be variability with time-scales of the order of t_{rl} in case (iii). Although there is little information about the Z-dependence of the ion work function E_c, we can be confident that it must ultimately decrease to a very small value in the limit $Z \rightarrow 1$. Thus there exists a critical value of Z at which the boundary condition $\mathbf{E}_{\parallel} \neq 0$ fails locally in some region of the polar cap and this must ultimately be reached as a consequence of proton emission. But the acceleration field is a function of the condition of the whole polar cap and the overall level of complexity is such that we make no attempt here to proceed further with this analysis except to note that time-independent case (iii) solutions are not possible.

4.2 Factors relevant to short time-scale variability

There are several characteristic times that are important factors in examining the possibility of short time-scale variability of plasma production in cases (ii) and (iii). The first is the proton diffusion time t_p derived from z_p and D. Then the initial e-folding growth time for a curvature-radiation electron-positron cascade must be of the order of $t_{ee} = 2l_a/c$, where l_a is the open- magnetosphere acceleration length interval concerned. The third parameter is t_{cond} the time-scale for thermal diffusion from the shower maximum z_p to the surface.

For an estimate of t_{cond}, we require the surface temperature fluctuation from T to $T + \delta T(0,t)$ caused by a fluctuation $\delta X(z', t')$ in the thermal power input at depth $z' < 0$ below the polar-cap surface at time t'. The correct Green function for this particular problem is cumbersome (see Carslaw & Jaeger 1959), but the characteristic value of $t - t'$ is, for present purposes, satisfactorily approximated by $t_{cond} = Cz_p^2/2\lambda_{\parallel}$. Here $\lambda_{\parallel}$ is the longitudinal coefficient of thermal conductivity and C is the specific heat of surface matter, for which an upper limit given by the Dulong and Petit law is assumed. Transport coefficients at $B > B_c$ in neutron-star envelopes and crusts have been calculated by Potekhin (1999). The longitudinal coefficient of thermal conductivity is not a particularly rapidly varying function of B and T and, at $Z = 26$, $B = 3B_c$ and $T = 10^6$ K, is approximately 6×10^{14} erg cm^{-1} s^{-1} K^{-1}. With the ion number density given by equation (4) and a shower depth of $z_p = 10l_r$, the time is $t_{cond} \approx 5\times 10^{-10}$ s, which is short compared with t_{ee} and with the estimates of t_p obtained from equations (21) - (24). It is also possible to confirm that, for the expected power inputs associated with the derived temperature limit given by equation (33), the temperature gradient between the shower maximum and the surface is no more than of the order of 10^6 K cm^{-1}, which is negligible.

Thus in relation to variability on time-scales set by t_{ee} or t_p, we can assume that the surface temperature fluctuations are instantaneous functions of the reverse-electron power input to the polar-cap surface. This is one of three important factors. The second is the mass-number fractionation of any atmosphere which was described in Section 3.2. The final factor is that the accelerated protons produce a quite negligible reverse-electron flux, as can be seen by examining their electromagnetic interactions with the thermal radiation field. Protons accelerated to energies of the order of 10^3 GeV are well below the threshold for pair-creation, directly or by Compton scattering, and in any case the cross-sections concerned are small. However, accelerated ions are also below the pair-creation threshold but, as we have seen in Section 3.3, those with Z sufficiently high to have bound electrons produce a reverse-electron flux by a sequence of photo-dissociation reactions with blackbody photons which increase the ion charge from $\tilde{Z}$ toward Z (Jones 1981; see Medin & Lai 2006 for atomic ionization energies at $B \approx B_c$).

Possible sources of the electron-positron current density J_e in neutron stars with $\mathbf{\Omega} \cdot \mathbf{B} < 0$ need consideration and we refer to the recent review of Harding & Lai (2006). The growth of curvature radiation cascades requires a large $E_{\parallel}$, that is, a partial vacuum above the polar cap ($J < J_{crit}$), with the possibility that $J_{crit} = 0$. A further source is inverse Compton scattering of the blackbody radiation field by the reverse electron flux from accelerated ions, followed by magnetic conversion of the scattered photons. This was mentioned briefly at the end of Section 3.3 and we refer to Harding & Muslimov (2002) for further details.

4.3 Short time-scale variability; $E_{\parallel} = 0$

A first investigation of short time-scale variability in case (ii) with $T_{max} > T_c$ is most easily made, as in Section 4.1, with the assumption of a time-independent J, with negligible J_e. The proton current is then,

$$J_p(t) = K \int_{-\infty}^{t} dt' f_p(t - t') \left(J - J_p(t')\right), \tag{37}$$

where here the parameter K is a constant given by equations (31) or (32). The time t in this Section is dimensionless and in units of t_p. Equation (37), with J determined by solution of equation (1) for the immutable boundary condition $\Phi = 0$ on the surface separating open from closed magnetic flux lines, may not be a completely adequate description of short time-scale variability of the system, but is a useful starting point. It has the obvious time-independent solution $J_p = KJ/(K + 1)$ but it can be seen that for any small fluctuation in proton current density, equation (37) reduces to a homogeneous Volterra equation of the second kind having no square-integrable non-zero solution. For the present problem as opposed to that of Section 4.1, we choose to introduce a short time-scale fluctuation as an inhomogeneous term $\delta J_{p0}(t)$ centred on $t = 0$, and rewrite the equation as,

$$J_p(t) = KJ + \delta J_{p0}(t) - K \int_{0}^{\infty} d\tau f_p(\tau) J_p(t - \tau), \tag{38}$$

with $\tau = t - t'$. We can see that solutions that are not square-integrable exist for $K > 1$. For example, the case $f_p(\tau) = \delta(\tau - \tau_0)$ gives the obvious iterative solution,

$$J_p(t) = \frac{K}{K+1}J + \delta J_{p0}(t) - K\delta J_{p0}(t - \tau_0) + \\ K^2\delta J_{p0}(t - 2\tau_0) - .... \tag{39}$$

More generally, we can introduce the Fourier transform of J_{p0} into the formal iterative solution of equation (38) and sum to infinity to obtain,

$$J_p(t) = \frac{K}{K+1}J + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\delta J_{p0}(\omega)\exp(-i\omega t)}{1 + Kf_p(\omega)}, \tag{40}$$

where,

$$f_p(\omega) = \int_0^{\infty} d\tau f_p(\tau)\exp(i\omega\tau). \tag{41}$$

As expected, the trivial case of equation (38) can be recovered by summation over the residues of those poles in the integrand that are in the upper half of the complex-ω plane. On the other hand, there exist simple functions, such as $f_p(\tau) = a\exp(-a\tau)$ with constant a that give no such poles. This indicates that the behaviour of $f_p(\tau)$ at small $\tau > 0$ is crucial, so that for non-trivial functions, such as equation (23), there appears to be no alternative to a numerical search for zeros in the denominator. Thus for f_p given by equation (23) we find, for $K = 100$, poles at $\omega = \pm 4.85 + 1.88i$. For smaller K they approach, but do not reach, the real axis; for $K = 25$, the poles are at $\omega = \pm 3.4 + 0.1i$.

Our conclusion is that there is no doubt that the possibility of instability exists for large K with exponential growth of short time-scale fluctuations until the condition $0 < J_p < J$ is no longer satisfied and equation (37) fails. But unfortunately, certainty is not possible because we do not have sufficient knowledge of the actual shape of the formation depth distribution g_p at depths $z_p < z < 0$ but close to the surface. The existence of poles in the upper half of the complex-ω plane in equation (40) appears to depend crucially on the time interval between the creation of protons and their arrival at the surface. This almost certainly depends on the LPM effect considered in Section 2.4.

The behaviour of the system when equation (37) fails is easy to see. Suppose, for example that $T_{res} > T_c$, as must be the case in many pulsars, so that the boundary condition $\mathbf{E}_{\parallel} = 0$ is always maintained, also that $J_{crit} < J$ so that growth of curvature radiation is not possible. Instability resulting in excess proton production leads to the formation of a proton atmosphere at $z > z_1$, as described in Section 3.2, and the surplus results in a current density containing only the proton component of magnitude $J_p = J$. For the interval of time within which this state persists there is no reverse-electron flux, no further proton formation, and the surface temperature falls promptly to T_{res} in a time no more than an order of magnitude greater than t_{cond}. At the end of this interval, the proton atmosphere is exhausted and J_p falls abruptly to a value given by the tail of the distribution of f_p, much below its time-independent value, $KJ/(K+1)$. Consequently, there will be a prompt burst of ion acceleration and of outward moving positrons produced by the inverse Compton scattering process (see Harding & Muslimov 2002) unless $T_{res} < T_c$. Owing to the time-delay, this can be expected to lead to the formation of a further proton excess. Strict periodicity is not to be expected because, as in the case of medium time-scale variability, there is no obvious reason why the state of a very thin layer, of height $z \sim z_1$ should be uniform over the whole polar-cap area.

4.4 Short time-scale variability; $\mathbf{E}_{\parallel} \neq 0$

In a time-independent state, this boundary condition requires $T_{max} < T_c$, where T_{max} is the polar-cap blackbody temperature given by equation (33). But in a state of short time-scale variability, it can be satisfied intermittently provided the less restrictive condition $T_{res} < T_c$ is valid. We can begin by considering a steady state with $J > J_{crit}$ as in the previous section, with polar-cap temperature $T > T_c$. The instability leads to a state of excess proton production and to a very prompt decrease in temperature to T_{res} in a time probably one or two orders of magnitude larger than t_{cond} but still much smaller than t_p and to a similarly fast partial collapse of the ion LTE atmosphere, so that only protons and any low-Z ions remain. When these are exhausted, the current density J promptly decreases toward zero and if J_{crit} has no finite value (the star cannot support pair creation by curvature radiation at any current density) it would appear that plasma generation ceases. But a finite J_{crit} allows pair creation when $J < J_{crit}$. Thus $J_e + J_Z$ increases rapidly and a further proton excess is generated.

The conclusion to be drawn here is that, given short time-scale variability, the important polar-cap surface temperature is T_{res} which should be compared with the work-function related temperature $T_c = 5.6 \times 10^4 B_{12}^{0.7}$ K. Thus for $T_{res} = 5 \times 10^5$ K, a surface magnetic flux density $B_{12} > 23$ is required for the boundary condition $\mathbf{E}_{\parallel} \neq 0$ to be satisfied.

5 CONCLUSIONS

There seems to be no known reason why neutron stars with $\mathbf{\Omega} \cdot \mathbf{B} < 0$ and hence positive polar-cap corotational charge density should not be formed at a rate of the same order as those with the opposite sign. The purpose of this paper has been to investigate some of the features of plasma formation at the polar cap to see if these correlate with the properties of any subset of the observed population of radio pulsars. Specifically, we have attempted to find the formation rate for protons in the electromagnetic showers produced by the reverse electron flux at the polar cap. These protons diffuse rapidly to the surface and, having both a negligibly small work function and the highest charge to mass ratio, are preferentially accelerated upward by the electric field component parallel with $\mathbf{B}$. Proton formation is necessarily accompanied by evolution of the nuclear charge within the very thin surface layer in which the showers occur.

The proton formation rate was estimated previously (Jones 1981) at magnetic flux densities of $B_{12} \sim 1$ but the present paper has been addressed to the problem at much higher fields ($B \sim B_c$) owing to the realization that such values, though infrequent, are not unknown in the distribution of inferred dipole-field strengths. Additionally, an important subset, those radio pulsars exhibiting the nulling phenomenon, have higher than average fields, as noted at the beginning of Section 4. Values of the parameter K, defined as the number of protons formed per unit positive ac-

celerated nuclear charge, are given in Table 1 for $B \sim B_c$, but its parametrization in equation (31) is not valid at small values of B for which its asymptotic value K_{as} is given by equation (32).

Proton formation and the evolution of nuclear charge have the consequence that particle fluxes under the case (ii) boundary condition defined in Section 1 display much more complexity than might have been supposed. In particular, there is an instability, given by equation (36), which is realized in almost any radio pulsar, giving time variability typically over intervals of the order of t_{rl}. This is the time in which one radiation length of matter flows from the polar cap and is within the orders of magnitude associated with nulling intervals. Owing to the proton formation-depth distribution, fluctuations in the surface nuclear charge grow until the system alternates between having high values close to $Z(-\infty)$ and nuclear charges so low that the reverse electron flux is negligible. The latter state has no source of electron-positron pair creation at the polar cap and may possibly be associated with the intervals of null emission. But the state of the whole polar cap is almost certainly much more complex than a simple transition between high and low surface nuclear charge. The total depth of the shower formation region and of the LTE atmosphere is very many orders of magnitude smaller than the polar cap radius $u_0 \sim 10^4$ cm. The question then is why the time-variable state of the surface and atmosphere should be in phase over the whole polar-cap area. No serious attempt has been made here to investigate this problem but there seems to be no reason to expect such phase uniformity. Chaotic behaviour appears more probable. Similar time-variability and complexity are expected in case (iii).

Instability leading to short time-scale variability is also possible. The relevant unit of time is t_p, the characteristic time for proton diffusion from depth z_p to the surface. Here, growth of the instability results in alternation between two states of which the first is an accelerated proton current density $J_p = J$ with $J_e = J_Z = 0$, no reverse electron energy flux and a polar-cap surface temperature equal to T_{res}. Abrupt exhaustion of the proton atmosphere then allows rapid growth in J_e and J_Z and the formation of a short pulse of electron-positron pair formation. This instabilty appears to be common to both cases (ii) and (iii). It is also interesting because it defines the temperature at which the case (iii) boundary condition $\mathbf{E}_{\parallel} \neq 0$ must be satisfied. This must be T_{res} and not the temperature T_{max} of equation (33) defined by the time-averaged maximum reverse-electron energy flux. The condition is then $T_{res} < T_c$. From our parametrization of the work functions found by Medin & Lai (2006), the critical temperature is $T_c = 5.6 \times 10^4 B_{12}^{0.7}$ K, and we find that the required polar-cap surface magnetic flux density is $B_{12} > 2.3 T_{res5}^{1.4}$ in which T_{res5} is the polar-cap temperature, in units of 10^5 K, in the absence of a reverse-electron energy flux. This unit of T is not inappropriate for the whole-surface temperatures of the older radio pulsars and the result shows that the case (iii) boundary condition can be realized at much lower magnetic flux densities than was previously thought. At the same time, the similarities between the instabilities found in cases (ii) and (iii) suggests that, in terms of observable phenomena, the distinction between the two boundary conditions may not be of great significance. It is to be emphasized that, in both cases, the cessation of ion flow is not temperature-controlled (by the ionic analogue of Richardson's formula). Rather, it is controlled by excess proton production and by the consequent formation of a proton atmosphere at $z > z_1$. In case (iii), only the onset of ion flow is a result of surface heating.

An interesting state is that of case (ii) with the parameter $K < 1$ derived from the $B_{12} \ll 1$ expression for the asymptotic limit given by equation (32). This can be realized at polar-cap temperatures greater than 10^6 K, particularly if this temperature is supported by a large value of T_{res}. With reference to equation (36), we can see the associated proton and ion current densities are stable and time-independent. Monoenergetic ion beams do interact with an electron-positron plasma (see Cheng & Ruderman 1980) but we anticipate little if any electron-positron pair production by inverse Compton scattering at $B_{12} \ll 1$ and there may be no observable consequences of such a system. This may be particularly relevant to young neutron stars or to any older neutron stars that for other reasons have very high values of T_{res}.

A further comment can be made about the reverse electron energy flux from any outer gap that exists. It is hard to make quantitative estimates of this, but there is every possibility that the consequent rate of proton formation would completely inactivate polar-cap pair formation as in nulling intervals.

The time-variabilities considered so far have been found assuming a time-independent boundary condition $\Phi = 0$ on the surface separating open from closed magnetic flux lines. But we have not addressed the problem of how, or if, this boundary condition is maintained. The proton diffusion time t_p is little more than an order of magnitude greater than the typical transit time of order $u_0 c^{-1}$ and a correct treatment of polar-cap phenomena may require not just the solution of equation (1) but solution of the full set of Maxwell equations with retardation. Thus we do not exclude the existence of further instabilities beyond those considered here. This comment may also be relevant to case (i) in which the corotational charge density is negative. The considerations of this paper have no application to this class of neutron star. The electron work function is so small that there is no doubt the condition $\mathbf{E}_{\parallel} = 0$ is satisfied at all times on the polar cap, also that electrons are freely available to maintain the $\Phi = 0$ boundary condition on the surface separating open from closed magnetic flux lines.

It would be of interest to know how boundary conditions (i), (ii) and (iii) are distributed within the observed population of radio pulsars and neutron stars and whether or not the short time-scale temporal variability shown here to be a feature of cases (ii) and (iii) contributes to the observed microstucture of some radio pulses. Perhaps systematic measurements of individual pulse structures with $10^{-6} - 10^{-5}$ s resolution would provide some insight.

REFERENCES

Assafiri Y. I., Thompson M. N., 1986, Nucl. Phys. A, 460, 455

Bethe H. A., 1934, Proc. Camb. Phil. Soc., 30, 524

Carslaw H. S., Jaeger J. C., 1959, Conduction of Heat in Solids, Oxford University Press, p. 358

Cheng A. F., Ruderman M. A., 1980, ApJ, 235, 576
Erber T., 1966, Rev. Mod. Phys., 38, 626
Fultz B. C., Alvarez R. A., Berman B. L., Meyer P., 1974, Phys. Rev. C, 10, 608
Gil J., Haberl F., Melikidze G., Geppert U., Zhang B., Melikidze G., Jr., 2008, ApJ, 686, 497
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Gonthier P. L., Harding A. K., Baring M. G., Costello R. M., Mercer C. L., 2000, ApJ, 540, 907
Hansen H. D., et al, 2004, Phys. Rev. D, 69, 032001
Harding A. K., Lai D., 2006, Rep. Progr. Phys., 69, 2631
Harding A. K., Muslimov A. G., 2002, ApJ, 568, 862
Hayward E., 1963, Rev. Mod. Phys., 35, 324
Jones P. B., 1979, ApJ, 228, 536
Jones P. B., 1981, MNRAS, 197, 1103
Jones P. B., 1985, Phys. Rev. Lett., 55, 1338
Jones P. B., 1986, MNRAS, 222, 577
Kerkhove E., et al, 1985, Phys. Rev. C, 31, 1071; Phys. Rev. C, 32, 368
Klein S., 1999, Rev. Mod. Phys., 71, 1501
Kramer M., Lyne A. G., O'Brien J. T., Jordan C. A., Lorimer D. R., 2006, Sci., 312, 549
Landau L., Rumer G., 1938, Proc. Roy. Soc., A166, 213
Lange Ch., Kramer M., Wielebinski R., Jessner A., 1998, A&A, 332, 111
Latal H. G., 1986, ApJ, 309, 372
Manchester R. N., Hobbs G. B., Teoh A., Hobbs M., 2005, Astron. J., 129, 1993
Medin Z., Lai D., 2006, Phys. Rev. A, 74, 062507, 062508
Melrose D. B., 1995, J. Astrophys. Astr., 16, 137
Michel F. C., 1974, ApJ, 192, 713
Muslimov A. G., Harding A. K., 1997, ApJ, 485, 735
Muslimov A. G., Tsygan A. I., 1992, MNRAS, 255, 61
Nordheim L. W., Hebb M. H., 1939, Phys. Rev., 56, 494
O'Keefe G. J., Thompson M. N., Assafiri Y. I., Pywell R. E., Shoda K., 1987, Nucl. Phys. A, 469, 239
Potekhin A. Y., 1999, A&A, 351, 787
Potekhin A. Y., Yakovlev D. G., Chabrier G., Gnedin O. Y., 2003, ApJ, 594, 404
Ruderman M. A., Sutherland P. G., 1975, ApJ, 196, 51
Semionova L., Leahy D., 2001, A&A, 373, 272
Slattery W. L., Doolen D. G., DeWitt H. E., 1980, Phys. Rev. A, 21, 2087
Smirnova T. V., Tul'bashev S. A., Boriakoff V., 1994, A&A, 286, 807
Wang N., Manchester R. N., Johnston S., 2007, MNRAS, 377, 1383

This paper has been typeset from a TEX/ LATEX file prepared by the author.